Likelihood Estimation of Jump-Diffusions

Extensions from Diffusions to Jump-Diffusions, Implementation with Automatic Differentiation, and Applications

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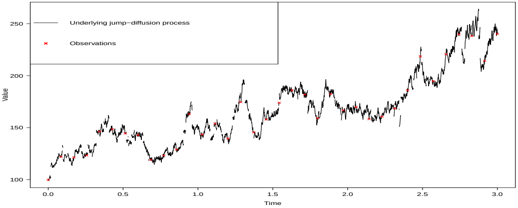
Outline

Motivation

- How does financial bubbles develop?

The problem



- Continuous time process with an infinitesimal description

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t + c(t, X_{t^-}, \xi_{N_{t^-}+1})dN_t, \quad (1)$$

but discrete observations.

- How to find the transition density to build the (log)likelihood:

$$l(\theta|x_{t_1}, \ldots, x_{t_n}) = \sum_{i=2}^{n} \log p(x_{t_i}|x_{t_{i-1}}, \theta)? \quad (2)$$

The solution

- Replace the continuous process with a discrete approximation where it is possible to find the transition density.
- The method of Preston & Wood 2012 for time-homogeneous diffusions:
  1. Develop an Itô-Taylor expansion of the sample path.
  2. Calculate the moment generating function of the retained terms in the expansion.
  3. Approximate the inverse Fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(s)e^{-isx}ds. \tag{3}$$

- We extended this method to make it applicable to time-homogeneous jump-diffusions of the following form:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + c(\xi_{N_{t-}+1})dN_t. \tag{4}$$

ITSPA

Let X_t be a jump-diffusion, where $Y_t = \sum_{i=1}^{N_t} Z_i$, and let $\widetilde{X}_{t-}$ denote the approximate solution to the pure diffusion, based on a discretization scheme.

Theorem (ITSPA)

An approximation to the transition density of X_t follows from the saddlepoint approximation to the transition density of the approximated process $\widetilde{X}_t = \widetilde{X}_{t-} + Y_t$, which we call the Itô-Taylor saddlepoint approximation:

$$f_{X_t}(x) \approx spa\left(f_{\widetilde{X}_t}; x\right) = \frac{\exp\left\{K_{\widetilde{X}_t}(\hat{s}) - \hat{s}x\right\}}{\sqrt{2\pi K''_{\widetilde{X}_t}(\hat{s})}}, \quad (5)$$

where

$$K_{\widetilde{X}_t}(\hat{s}) = K_{\widetilde{X}_{t-}}(\hat{s}) + K_{Y_t}(\hat{s}) = K_{\widetilde{X}_{t-}}(\hat{s}) + \lambda t\left(M_Z - 1\right), \quad (6)$$

and M_Z is the MGF of the iid jump magnitudes and $\hat{s}$ the saddlepoint.

mITSPA

Theorem

The Itô-Taylor saddlepoint approximation mixture (mITSPA) to the transition density of X_t is as follows:

$$mspa\left(f_{\widetilde{X}_t};x\right) = spa\left(f_{\widetilde{X}_{t-}};x\right)e^{-\lambda t} + spa\left(f_{\widetilde{X}^*_t};x\right)\left(1-e^{-\lambda t}\right), \tag{7}$$

where $\widetilde{X}^*_t = \widetilde{X}_{t-} + \sum_{i=1}^{N^*_t} Z_i$, and N^*_t has a zero-truncated Poisson distribution with intensity λt and defined by $N^*_t = N_t | N_t > 0$. The related CGF of the compounded zero-truncated Poisson process Y^* is given by:

$$K_{Y^*}(s) = \lambda t\left(M_Z(s) - 1\right) + \log\left(1 - e^{-\lambda t M_Z(s)}\right) - \log\left(1 - e^{-\lambda t}\right). \tag{8}$$

Fourier-Gauss-Laguerre

Theorem (Fourier-Gauss-Laguerre)

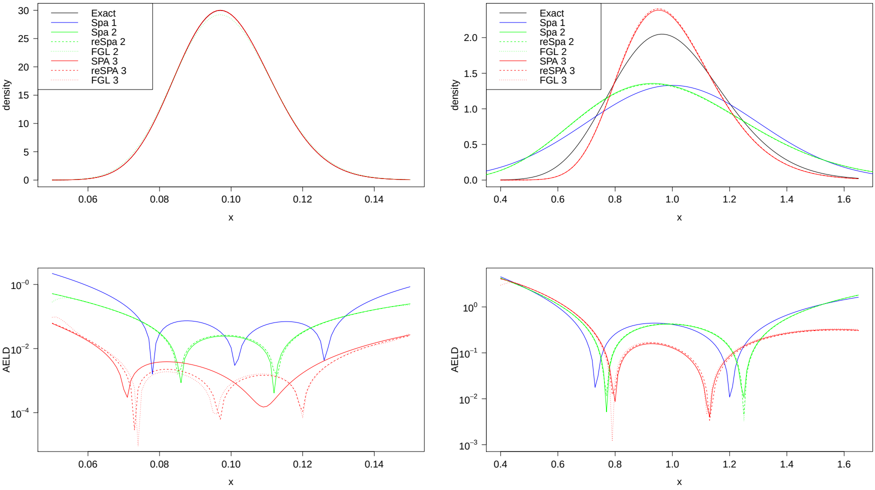
The Fourier-Gauss-Laguerre (FGL) approximation to the transition density of X_t is given by:

$$fgl\left(f_{\widetilde{X}_t}; x\right) = \frac{1}{\pi} \sum_{j=1}^{n} w_j \Re\left(\phi_{\widetilde{X}_t}(s_j) e^{s_j - i s_j x}\right), \tag{9}$$

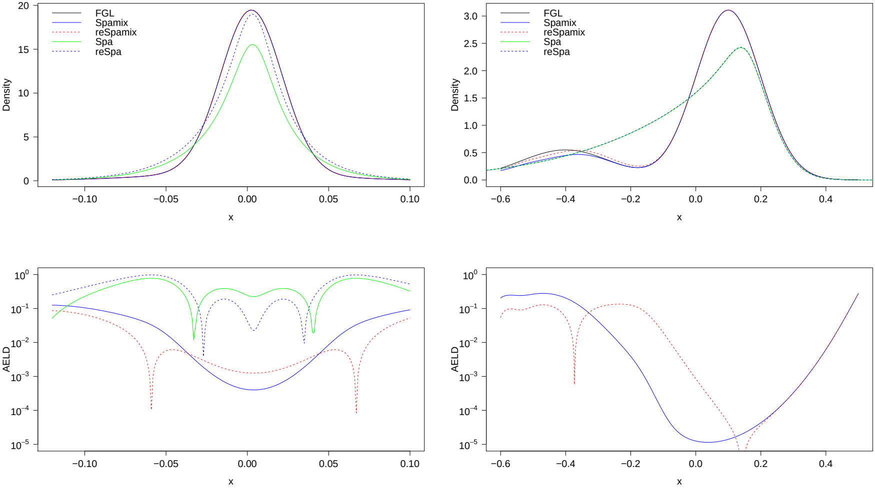
where w_j and s_j are the weights and the abscissa respectively in the Gauss-Laguerre method of order n. The characteristic function is found by multiplying the characteristic function for one of the discretizations $\widetilde{X}_t^-$ of the diffusion part and the characteristic function for the compounded Poisson process Y_t:

$$\phi_{\widetilde{X}_t}(s) = \phi_{\widetilde{X}_{t-}}(s)\phi_{Y_t}(s). \tag{10}$$

The Cox-Ingersoll-Ross process



The Merton jump-diffusion



Results from MLE

Results from likelihood-based inference for processes with known solutions (GBM, OU, CIR, MJD).

For pure diffusions:

- All the methods and schemes produce good results compared to the estimates based on using the exact transition densities.
- Renormalization does not seem to have a large and beneficial effect.
- The saddlepoint approximation approximates the transition density accurately.

For jump-diffusions:

- Renormalization of the mITSPA seems to be necessary both for parameter estimates and especially the value of the likelihood.

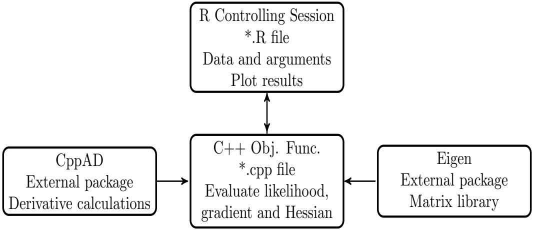
A note on speed

Method	l_θ	$\nabla_\theta l$	$\mathbb{H}_\theta$
	CIR		
ITSPA			
scheme 1	1.888	7.196	51.400
scheme 2	3.182	11.483	73.114
scheme 3	4.689	15.445	95.580
reITSPA			
scheme 2	137.473	509.000	6298.884
scheme 3	174.192	607.021	4563.417
FGL			
scheme 2	7.489	16.445	78.742
scheme 3	24.666	57.116	273.790
	MJD		
mITSPA	9.488	35.574	396.939
remITSPA	151.196	541.689	12120.45
FGL	16.336	36.031	219.360

Automatic differentiation and TMB

Implementation with automatic differentiation (AD) using the TMB package for R and C++.

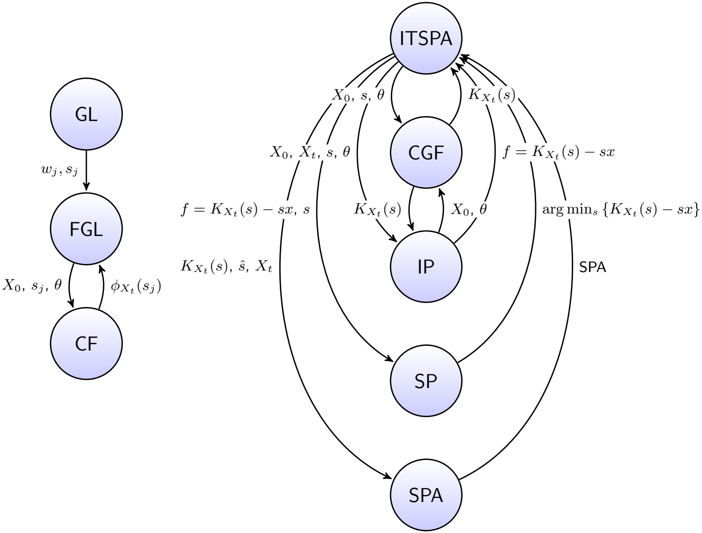
- Given a computer algorithm defining a function, AD is a set of techniques used to evaluate numerically the derivatives of that function.
- Based on the property of programming languages such as C++ of decomposing expressions into elementary operations.
- Can be implemented using operator overloading.



Benefits and extensions

- It is possible to utilize AD inside the C++ part of the program:
  1. Evaluation of $K''_X(\hat{s})$ in the expression of the saddlepoint approximation.
  2. Makes it possible to solve the inner problem: $K'_X(\hat{s}) = x$.
  3. Comes in handy when calculating moments which was needed for renormalization.
- We extended TMB with the following:
  1. Modified Bessel function of the first kind (drawn from R).
  2. Log-normal density function (drawn from R).
  3. A templated complex data type, `cType`.

Complexity of methods



Conclusion on methods

- For diffusion processes, we suggest using the ITSPA methods on the basis of stability and speed.
- For jump-diffusions, we suggest using the FGL method on the basis of accuracy, stability and speed.

Application: Analysis of stock prices as nonlinear processes

- Applied the (ITSPA and FGL) methods to the question of nonlinearity and jumps as significant additions to stock price models.

Model name	Drift component	Diffusion component	Jump component
GBM	rS_t	σS_t	None
CEV	rS_t	σS_t^α	None
nlModel 1	rS_t^α	σS_t^α	None
nlModel 2	rS_t^α	σS_t^β	None
MJD	$(r - \lambda\hat{k})S_t$	σS_t	Log-normal
CEVJD	$(r - \lambda\hat{k})S_t$	σS_t^α	Log-normal

Results on nonlinearity and jumps

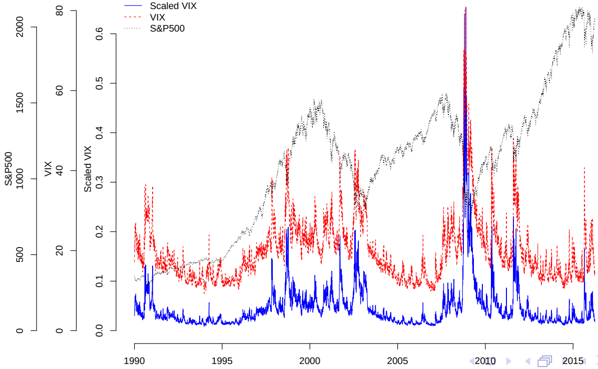
- Using daily logarithmic stock quotes
  1. Shanghai Securities Exchange (SSE) from 03.01.2005 until 16.10.2007
  2. Dow Jones Industrial Average (DJIA) from 29.04.1925 until 03.09.1929
  3. Standard & Poors 500 (S&P500) from 11.10.1990 until 24.03.2000
- Both the addition of nonlinearity and of jumps are significant improvements.
- The α parameter in nlModel 2 is not a significant addition to the model.
- The CEVJD model is a significant improvement relative to the MJD model.

SSE MLE results

Model		Parameters							Statistics		
		r	σ	α	β	λ	μ	ν	$l(\hat{\theta}; \mathbf{x})$	D	p-value
						SSE bubble of 07					
GBM	est	0.6268	0.2629						1796.7		0.0013
	se	0.1604	0.0071								
CEV	est	0.4718	0.0118	1.4072					1826.2	59	0.0011
	se	0.1478	0.0021	0.0244							
nlModel 1	est	0.0249	0.0112	1.4132					1827.6	61.8	0.0030
	se	0.0082	0.0019	0.0235							
nlModel 2	est	0.0001	0.0120	2.0744	1.4046				1828.9	64.4	0.0020
	se	0.0005	0.0022	0.3920	0.0247						
MJD	est	0.6261	0.1694			92.1	-0.0039	0.0202	1840.9	88.4	0.7645
	se	0.1584	0.0272			64.9	0.0027	0.0051			
CEVJD	est	0.4356	0.0128	1.3769		13.6	-0.0094	0.0345	1851.8	110.2	0.0179
	se	0.1525	0.0033	0.0345		8.2	0.0086	0.0085			

Application: stochastic volatility models

- Which stochastic volatility model is preferable?
- Transformed VIX indices from January 1990 to March 2016 as observations of implied volatility.
- Used the ITSPA with the Euler scheme (due to stability).



The results on stochastic volatility models

Model				Estimates				Statistic
Name	Drift	Diffusion		κ	α	σ	δ	$l(\hat{\theta}, \mathbf{x})$
OU	$\kappa(\alpha - V_t)$	σ	est	7.46228	0.04550	0.18001		20308
			se	0.76305	0.00469	0.00158		
CIR	$\kappa(\alpha - V_t)$	$\sigma\sqrt{V_t}$	est	3.34275	0.04543	0.49902		24585
			se	0.73762	0.00617	0.00433		
GARCH(1,1)	$\kappa(\alpha - V_t)$	σV_t	est	2.22330	0.05407	2.13335		26096
			se	0.81318	0.01295	0.01855		
3/2 model	$V_t(\alpha - \kappa V_t)$	$\sigma V_t^{\frac{3}{2}}$	est	86.37495	5.96746	12.26975		25646
			se	18.13629	0.65073	0.10669		
GMR	$\kappa(\alpha - V_t)$	σV_t^{δ}	est	2.19812	0.05449	2.99164	1.10223	26133
			se	0.84034	0.01386	0.12376	0.01202	

- Of the standard models, the continuous time GARCH(1,1) model seems preferable.

Application: A short-rate model

$$dr_t = \kappa(\alpha - r_t)dt + \sigma r_t^{\delta} dW_t + Y_t dN_t$$

κ	α	σ	δ	λ	μ	ν^2
0.001474	2.142	0.114	1.634	30	≈ 0	0.01

Further work

1. An extension of the mITSPA and the FGL methods to several dimensions.
2. An extension of the methods to a more general jump-diffusion process, where the jump part of the SDE may be allowed to take a more general form.
3. A more extensive study of nonlinearity in financial markets.
4. The study of a more general mean reverting jump-diffusion process as a model for stochastic volatility, an extension of the *basic affine jump-diffusion* process. E.g:

$$dV_t = \kappa(\alpha - V_t)dt + \sigma V_t^{\delta} dW_t + dJ_t, \tag{11}$$

where J_t is a compounded Poisson process with gamma distributed jumps.

Thank you